Problem 1. Identical interest games and potential games

Consider an N-player game, in which each player P_i has a set Γ_i of pure strategies. Let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_N$. A game is said to be of *identical interests* if all players have the same cost function, that is

$$J_i(\gamma) = \phi(\gamma) \quad \forall i = 1, \dots, N,\ \forall \gamma \in \Gamma.$$

a) Show that an identical interest game has a pure strategy Nash equilibrium. Furthermore, this Nash equilibrium is admissible.

b) Consider the identical interest game below. Show that it has two admissible Nash equilibria, but these equilibria are not interchangeable: $A = B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

c) Suppose γ is a Nash equilibrium for the game. Is it a minimizer of ϕ? Justify your answer.

Solution:

a) The function ϕ is an exact potential for the game since for all i, i.e.,

$$J_i(\gamma_i, \gamma_{-i}) - J_i(\gamma_i', \gamma_{-i}) = \phi(\gamma_i, \gamma_{-i}) - \phi(\gamma_i, \gamma_{-i}), \quad \forall \gamma_i, \gamma_i' \in \Gamma_i, \gamma_{-i} \in \Gamma_{-i}.$$

Since ϕ takes on finite values (its domain is finite), ϕ has a minimizer $\gamma \in \Gamma$. This can be verified as follows: let $\gamma^* = \arg\min_{\gamma \in \Gamma} \phi(\gamma)$, that is, $\phi(\gamma^*) \leq \phi(\gamma)$. Hence, for all i, $\phi(\gamma_i^*, \gamma_{-i}^*) \leq \phi(\gamma_i, \gamma_{-i}^*)$ for all $\gamma_i \in \Gamma_i$. From the fact that ϕ is a potential function, it follows that $J_i(\gamma_i^*, \gamma_{-i}^*) \leq J_i(\gamma_i, \gamma_{-i}^*)$ for all $\gamma_i \in \Gamma_i$ and hence, γ^* is an admissible pure Nash equilibrium.

b) There are two pure Nash equilibria: (x, x) and (y, y). The two Nash equilibria are admissible, because they have the same cost for both agents, but they are not interchangeable, because the strategies (x, y) and (y, x) are not Nash equilibria.

c) No, a Nash equilibrium of the game is in general not a minimizer of ϕ. Consider for example, $\Gamma_i = \{1, 2\}$, $i = 1, 2$ and $\phi(1, 1) = -2, \phi(2, 2) = -1, \phi(1, 2) = \phi(2, 1) = 0$. The Nash equilibria are the strategies $(1, 1)$ and $(2, 2)$ but only $(1, 1)$ is the minimizer of ϕ.

Problem 2.

Provide two approaches for verifying whether the prisoner's dilemma game discussed in Lecture 01 is a potential game. Show your work for both approaches.

	Confess	Stay silent
Confess	(5, 5)	(0, 10)
Stay silent	(10, 0)	(1, 1)

Solution:

Approach 1:

To derive the exact potential function of the prisoner's dilemma, we solve the following set of equations:

$$\begin{aligned} J_1(C, C) - J_1(S, C) &= P(C, C) - P(S, C) \\ J_2(C, C) - J_2(C, S) &= P(C, C) - P(C, S) \\ J_1(S, S) - J_1(C, S) &= P(S, S) - P(C, S) \\ J_2(S, S) - J_2(S, C) &= P(S, S) - P(S, C) \end{aligned}$$

Set $P(C,C) = 0$. Then,

$$
\begin{aligned}
P(S,C) &= P(C,C) - J_1(C,C) + J_1(S,C) = 0 - 5 + 10 = 5\\
P(C,S) &= P(C,C) - J_2(C,C) + J_2(C,S) = 0 - 5 + 10 = 5\\
P(S,S) &= P(C,S) + J_1(S,S) - J_1(C,S) = 5 + 1 - 0 = 6
\end{aligned}
$$

Thus, potential function P is given by:

$$
P = \begin{bmatrix} 0 & 5 \\ 5 & 6 \end{bmatrix}.
$$

Approach 2:

We can check if the improvement along a simple closed path of length 4 is equal to zero. If yes, the game is potential. There are two possible simple closed path of length 4:

$$
\begin{aligned}
\mathcal{P}_1 = \{ &(\text{Confess}, \text{Confess}), (\text{Stay silent}, \text{Confess}),\\
&(\text{Stay silent}, \text{Stay silent}), (\text{Confess}, \text{Stay silent}),\\
&(\text{Stay silent}, \text{Confess})\}\\
\mathcal{P}_2 = \{ &(\text{Confess}, \text{Confess}), (\text{Confess}, \text{Stay silent}),\\
&(\text{Stay silent}, \text{Stay silent}), (\text{Stay silent}, \text{Confess}),\\
&(\text{Confess}, \text{Confess})\}
\end{aligned}
$$

The improvement for the first path is:

$$
\begin{aligned}
I(\mathcal{P}_1) := &\sum_{k=1}^{4} [J_{i_k}(\gamma(k)) - J_{i_k}(\gamma(k-1))]\\
= &J_1\,(\text{Stay silent}, \text{Confess}) - J_1\,(\text{Confess}, \text{Confess}) +\\
&J_2\,(\text{Stay silent}, \text{Stay silent}) - J_2\,(\text{Stay silent}, \text{Confess}) +\\
&J_1\,(\text{Confess}, \text{Stay silent}) - J_1\,(\text{Stay silent}, \text{Stay silent}) +\\
&J_2(\text{Confess}, \text{Confess}) - J_2(\text{Confess}, \text{Stay silent})\\
= &(10-5) + (1-0) + (0-1) + (5-10) = 5 + 1 - 1 - 5 = 0.
\end{aligned}
$$

The improvement for the second path can be computed in the same way and it is zero as well. Thus, the game is potential.